Proceedings of the American Academy of Arts and Sciences.

Vol. XLIV. No. 11. — March, 1909.

CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE THEORY OF BALLISTIC GALVANOMETERS OF LONG PERIOD.

By B. Osgood Peirce.

With a Plate.

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By B. Osgood Peirce.

Presented November 11, 1908. Received December 22, 1908.

If a ballistic galvanometer is to be used to measure the whole quantity of electricity which flows impulsively in a circuit when a condenser is discharged through it, or when the flux of magnetic induction through the circuit is suddenly changed, it can generally be assumed that the time during which the current lasts is so short that the flow practically ceases before the suspended system of the instrument has moved sensibly from its position of rest. That is, that the whole time of flow is not greater than, say, one fiftieth part of the time required for the needle or suspended coil to reach the end of its throw.

It is often desirable, however, to determine with accuracy the change of magnetic flux in a massive closed iron frame caused by a given change of excitation, and in such a case it usually happens that eddy currents in the metal or the inductance of the exciting coil so retard the change that the process lasts for a number of seconds at least. Under these circumstances a ballistic galvanometer of any ordinary form is practically useless. Indeed, according to the experiences [1] of Du Bois with such galvanometers as are to be found in most laboratories, the ballistic method fails when the time required for the change exceeds about one second.

Slow flux changes can be measured, nevertheless, with the aid of photographic records from a suitable oscillograph [2] either in the main circuit of the magnet or in the circuit of a testing coil wound about the iron. My experience with hundreds of such records seems to show,

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[1] Du Bois, The Magnetic Circuit in Theory and Practice, Atkinson's translation, § 216, London, 1896.

[2] T. Gray, Phil. Trans., **184** (1893); Thornton, Electrical Engineer, **29** (1902); Phil. Mag., **8** (1904); Electrician, 1903; Peirce, These Proceedings, **41** (1906); **43** (1907).

however, that the thickness of the photographed line obscures somewhat the slow changes when the exciting current has nearly reached its new value, and in the very sensitive instruments sometimes required for use in a secondary circuit there is a small but occasionally troublesome lag just at the beginning of the motion. For all ordinary purposes this method is wholly satisfactory if not always easy or convenient to carry out.

Such fluxmeters as I have been able to procure, though admirable in many ways, have not been so free from crawling, due apparently to the paramagnetic properties of their copper coils, that their indications can be trusted for very slow magnetic changes. If the fluxmeter coil [3] is not wound on a metal frame, the mutual damping caused by the action of currents in the coil, and the core which it surrounds, are not always effective unless the resistance of the exterior circuit is small, and this frequently makes an instrument which works very well for one piece of work, nearly useless for another.

When the excitation of the core of a large electromagnet initially in a given magnetic condition and under a given excitation is changed by a predetermined amount, it sometimes happens — as is well known — that the resulting change in the magnetic flux through the iron depends somewhat upon the manner in which the exciting current is changed; that is, the flux change is different when the current in the magnet coil is changed very gradually or in short steps from what it is when the change is made very suddenly. This difference is generally small, and seems to depend upon a variety of circumstances [4] in a way not yet very well understood, but it must be determined for every large magnet if the behavior of the core under given conditions is to be predicted with any great accuracy.

I have recently had occasion to inquire how the changes of magnetic flux in each of a number of large cores, of which two are represented by Figures A and B, corresponding to given changes in the current in the exciting coil, depend upon the manner of growth of that current, and since such oscillograph records as I was able to make were not wholly satisfactory for the purpose, I found it desirable to attempt to procure a ballistic galvanometer (preferably of the d'Arsonval type, to avoid disturbances due to changes in outside magnetic fields) of period so long that the throw of the coil due to a change of flux of the usual sort, lasting for say thirty seconds, should not be sensibly different

[3] Beattie, Electrician, Dec., 1902; Peirce, These Proceedings, **43** (1907).

[4] G. Wiedemann, Galvanismus, **3**, 738; Gumlich und Schmidt, Electrotechnische Zeitschrift, **21** (1900); Ruecker, Inaugural Dissertation, Halle, 1905; A. Hoyt Taylor, Phys. Rev., **23** (1906).

from the throw due to the same amount of electricity sent impulsively through the coil when at rest in its position of equilibrium.

The galvanometer I sought did not need to be very sensitive, but it must have one property which, according to my experience, is rare in suspended coil instruments; that is, there must not be the slightest

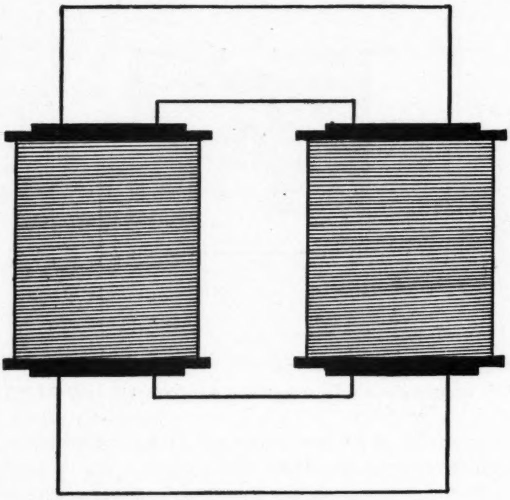

FIGURE A.

This electromagnet has a laminated core made of sheet iron one third of a millimeter thick and weighs about 300 kilograms.

sensible shift of the zero point due to thermal currents or to chemical action at the junctions when the galvanometer circuit should be closed on itself. This condition forbade the leading of the current into the galvanometer coil through the phosphor bronze or steel gimp by which the coil was suspended, and required that the whole galvanometer circuit, even to the binding posts and connectors, should be of one metal, copper.

It is of course not desirable to make the period of a ballistic galvanometer long by making the righting moment due to the suspending fibre small, for a weak fibre makes the zero point uncertain, and a large throw on one side usually shifts the zero point slightly in that direction unless the gimp is even stouter than that commonly used in sensitive

instruments. It seemed necessary, therefore, to increase the moment of the suspended system so much that in spite of a stiff suspending gimp the period should be long.

In the case of a galvanometer coil with a period several minutes long, it is difficult to tell by mere inspection for a few seconds whether the coil is really at rest at its zero or whether it has a very slight velocity

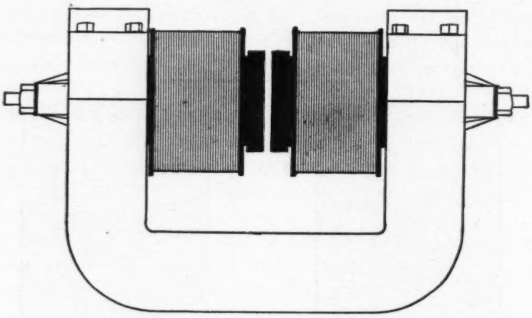

FIGURE B.

This magnet has a solid core which weighs about 1500 kilograms.

which in the course of its slow swing will lead to an addition of two or three millimeters to the amplitude of the throw. For this reason it was desirable that the coil should be subjected to some slight electromagnetic damping, though, as will appear later on, it was not possible to damp the coil critically.

The requirements enumerated above are so simple that it seemed at first that there would be no difficulty in meeting them all, and this would have been the case if it were not for the fact that the best copper and silver wire, and the best copper, silver, and aluminium sheet to be had in the market are usually so highly paramagnetic that in an intense magnetic field the galvanometer coil and the metal frame on which it is wound, if a frame be used, often acquire a large magnetic moment, and this increases in an irregular way the righting moment of the suspended system — perhaps to many times the value due to the gimp alone. The difficulty is an old one ; many persons have struggled with it, and some have succeeded in overcoming it more or less completely, by great care in the preparation of special wire for the purpose. The difficulties are, however, very much increased when it is necessary to provide a sufficient electromagnetic damping (air damping is some-

times objectionable) for a suspended system which in order to have the requisite moment of inertia must weigh perhaps 300 grams. Silk insulating material is generally magnetic, and so is most paraffine wax. A certain closed frame made by Mr. G. W. Thompson, the mechanician of the Jefferson Physical Laboratory, of the best obtainable sheet copper, to hold the coil of a d'Arsonval galvanometer of the common cored type, had a period of oscillation of about 2 minutes when suspended by a certain piece of gimp in free space, but a period of only 9 seconds when put in place in the instrument. In this case the righting moment due to the fibre was clearly wholly overshadowed by that due to the magnetism of the copper. When copper was wound on this frame, the magnetic moment of the whole, if placed between the poles of the permanent magnet, became so large that the whole suspended system could be deflected at will, when the circuit was open, by a bar magnet held in the hand outside the frame of the instrument.

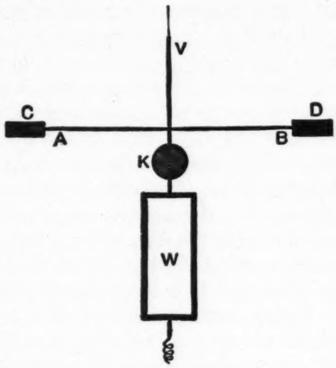

FIGURE C.

The horizontal rod AB is threaded, and the brass masses C, D can be screwed on the rod as far as is necessary. The system must be accurately balanced.

It is easy to make the period of an ordinary d'Arsonval galvanometer of the Ayrton and Mather form as long as, say, 120 seconds, by attaching two small brass masses symmetrically to the ends of a horizontal aluminium wire centred on the axis of suspension of the coil (Figure C), though it is not always easy to balance the coils and its weights so exactly that the throws shall be symmetrical on both sides of the zero point and that the instrument shall not be unpleasantly affected by changes of level. Galvanometers of this kind are often useful: several (one with a period of 158 seconds) have been used for years in the Jefferson Laboratory, and Professor A. Zeleny has lately employed a loaded coil galvanometer in his investigations of the properties of condensers. When the case of a d'Arsonval galvanometer is large enough, it is obviously better to load the coil with a disk centred on the axis of suspension than by several small masses, and in the instruments to be described in this paper thin disks with strongly reinforced rims were employed.

Two loaded coil d'Arsonval galvanometers have been constructed for me by Mr. Thompson. The first (V), shown in Figure 1, Plate 1, is about 76 centimeters high over all, and the gimp by which the coil is hung is 32 centimeters long. The brass disk, which is 11.4 centimeters in diameter, is rigidly attached to the rectangular frame (3 centimeters $\times$ 7 centimeters) upon which the copper wire coil is wound, and is accurately perpendicular to the axis of suspension.

After the copper frame constructed for this instrument had proved unsatisfactory, a cast type-metal frame was made to take its place. Of course this frame is not nearly so effective in damping the swings of the coil as a copper frame would be, but, on the other hand, its magnetic moment when it lies between the poles of the magnet of the galvanometer is not large. The insulated copper wire on the frame, however, gives a comparatively high moment to the whole suspended system, and the period of the galvanometer is much shorter — only about 140 seconds — than we supposed it would be with so large and heavy a disk. The binding posts and all the other connections are of copper, and the current is led into and out of the coil by two copper spirals under the disk, so fine that they do not exert any appreciable righting moment when the coil is deflected. The gimp is of steel, just stout enough to hold up securely the loaded coil.

The second galvanometer (W), represented in Figure 2, Plate 1, is about 111 centimeters high over all and 30 centimeters in diameter; the suspension gimp is about 80 centimeters long. It seemed nearly hopeless to attempt to get a sufficiently small righting moment with a hollow coil made of such wire as was to be obtained in the open market, so a coil of the Ayrton and Mather form was made for this instrument. The disk is accurately mounted on a metallic rod upon which the coil is fastened. The disk is built up of a thin sheet of flat aluminium with a brass rim about 24 centimeters in outside diameter and 15 millimeters in width. The current enters and leaves the coil through very fine copper spirals, one above and one below. If No. 44 or No. 46 copper wire be rolled out flat between jewellers' rolls or other similar device the resulting gimp serves to make a spiral which has extremely little torsional rigidity. It is possible to increase the number of field magnets in this instrument at pleasure. The logarithmic decrement of the galvanometer is small, but it has proved to be possible to bring the coil to rest at its zero point without much difficulty. The complete time of swing of the coil is about ten minutes, and the throws due to successive impulses of the same intensity agree with each other very closely indeed. I am much indebted to Mr. Thompson for the great skill and patience he has used in making these instruments. The apparatus was

mounted for use by Mr. John Coulson, who has helped me in all the work.

When the coil of a d'Arsonval galvanometer is disturbed from its position of equilibrium and is then allowed to swing under the action of a righting moment proportional to the angular deviation from its original place, the damping effects of the resistance of the air and of the induced currents in the frame and the coil, as they move between the poles of the permanent magnet of the instrument, may usually be accounted for, with an accuracy sufficient for most practical purposes, on the assumption that the motion of the suspended system is hindered at every instant by a force-couple of moment proportional to the angular velocity. Gauss and Weber showed that this hypothesis served to explain very well the motion of the magnets which they used in their measurements at Göttingen, and they put the mathematical theory of motion resisted in this way into the form in which it appears in most treatises on Physics [5] at the present day. When, however, a system swings under strong air damping, the motion sometimes [6] departs pretty widely from the Gaussian law at the beginning, at least, and it is not always safe to apply Gauss's equations to a ballistic galvanometer which has air damping as well as electromagnetic damping until one has found out whether the ratio of successive amplitudes is fairly constant during the whole motion, as Gauss's hypothesis demands. Even in the case of one of Gauss's own magnets, the logarithmic decrement of the amplitudes increased on a certain occasion from 1168×10^{-6} to

[5] Gauss, Resultate des Magnetischen Vereins, 1837; W. Weber, Resultate des Magnetischen Vereins, 1836, 1838; Maassbestimmungen, **2**; Math-phys. Abhandlungen der K. Sächs. Gesellschaft, 1852; Du Bois-Reymond, Monatsberichte der Berl. Acad., 1869, 1870; Chwolson, Bulletin de St. Petersbourg, 1881; Dorn, Ann. der Physik, **17** (1882); **35** (1888); Maxwell, Treatise on Electricity and Magnetism, **2**; G. Wiedemann, Lehre von der Elektricität, **3**; Deprez et d'Arsonval, Comptes Rendus, **94** (1882); Riecke, Abhandlungen der K. Gesellschaft der Wissenschaften zu Göttingen, **30**; Rachniewsky, Lumière Élect., **17** (1885); see also Lumière Élect., **29** (1888); **33** (1889); **45** (1892); Ledeboer, Comptes Rendus **102** (1886); Ayrton, Mather and Sumpner, Philosophical Magazine, **30** (1890); **42** (1896); **46** (1898); Classen, Electrotechnische Zeitschrift, **16** (1895); Sack, Electrotechnische Zeitschrift, **17** (1896); Des Coudres, Zeitschrift für Electrochemie, **3** (1897); Barus, Phys. Rev., **7** (1898); Salomon, Philosophical Magazine, **49** (1900); Robertson, Electrician, **46**, 901–904; **47**, 17–20 (1901); G. Kummell, Zeitschrift für Electrochemie, **7** (1901); Diesselhorst, Ann. der Physik, **9** (1902); Jaeger, Instrumentenkunde, 1903; Stewart, Phys. Rev., **16** (1903); White, Phys. Rev., **19** (1904); **22** (1906); **23** (1906); Shedd, Phys. Rev., **19** (1904); Smith, Phys. Rev., **22** (1906); A. Zeleny, Phys. Rev., **23** (1906); Wenner, Phys. Rev., **22** (1906); **25** (1907).

[6] Peirce, These Proceedings, **44** (1908).

1301×10^{-6} in 422 oscillations. It will appear in the sequel that the two long period galvanometers described in this paper follow the Gaussian law, if not exactly, still quite nearly enough to make it worth while to study their characteristics in the light of the usual theory.

The behavior of a damped ballistic galvanometer through which impulsive currents flow when the suspended system is away from its position of equilibrium and is in motion was first treated thoroughly by Dorn in a paper [7] written before d'Arsonval galvanometers were much used. In this paper Dorn studies the error introduced into observations made by Weber's methods of multiplication and of recoil, when the impulses are not properly timed. He also considers the case where the galvanometer is subjected to the action, not of a series of impulses, but of a continuous current, which lasts with given varying strength for a considerable time, and some of his equations have lately been put into other convenient forms by Diesselhorst. We shall find it desirable to derive from the beginning the special equations which we need in this paper.

The equation of motion of the coil of a d'Arsonval galvanometer, when the resisting moment is proportional to the angular velocity, is of the form

$$K \cdot \frac{d^2\theta}{dt^2} + 2\,a \cdot \frac{d\theta}{dt} + b^2\theta = 0, \tag{1}$$

where K is the moment of inertia of the suspended system about the axis of suspension. If this equation be written in the form

$$\frac{d^2\theta}{dt^2} + 2\,\alpha \cdot \frac{d\theta}{dt} + \beta^2\theta = 0, \tag{2}$$

α may be called the "damping coefficient," and β^2 the "restoring coefficient." It will be convenient to represent $d\theta/dt$ by ω, $(\beta^2 - \alpha^2)$ by ρ^2, and the complete time of swing of the coil by T.

If when $t = 0$, θ and ω have the given values θ' and ω', the general solution of (2) takes the form

$$\theta = e^{-\alpha t}\left[\theta' \cdot \cos \rho t + \frac{\omega' + \alpha\theta'}{\rho} \cdot \sin \rho t\right], \tag{3}$$

whence
$$\omega = e^{-\alpha t}\left[\omega' \cdot \cos \rho t - \frac{\alpha\omega' + \beta^2\theta'}{\rho} \cdot \sin \rho t\right]. \tag{4}$$

If, when the system is at rest in its position of equilibrium, an impulsive angular velocity ω_0 be given to it, and if after t_1 seconds have

[7] Dorn, Ann. der Physik, **17** (1882); Diesselhorst, Ann. der Physik, **9** (1902).

elapsed and the angular velocity has become ω_1, this velocity be impulsively increased by the amount ω_2, θ and ω are given during the first stage of the motion by the equations

$$\theta = \frac{\omega_0}{\rho} \cdot e^{-at} \cdot \sin \rho t, \tag{5}$$

$$\omega = e^{-at} \left[\omega_0 \cdot \cos \rho t - \frac{a\omega_0}{\rho} \cdot \sin \rho t\right], \tag{6}$$

and

$$\theta_1 = \frac{\omega_0}{\rho} \cdot e^{-at_1} \cdot \sin \rho t_1, \tag{7}$$

$$\omega_1 = e^{-at_1} \left[\omega_0 \cdot \cos \rho t_1 - \frac{a\omega_0}{\rho} \cdot \sin \rho t_1\right], \tag{8}$$

$$\rho = 2\,\pi/T,\ a = 2\,\lambda/T,\ a/\rho = \lambda/\pi,\ \beta^2 = \rho^2 + a^2.$$

If, then, for θ' and ω' in (3) and (4) we substitute θ_1 and ω_1 as given by (7) and (8), and for t in (3) and (4) put $(t - t_1)$, in order that the origin of time shall be that of (5) and (6), we shall get

$$\theta = \frac{\omega_0}{\rho}\, e^{-at} \cdot \sin \rho t + \frac{\omega_2}{\rho}\, e^{-a(t-t_1)} \sin \rho\,(t - t_1), \tag{9}$$

$$\omega = \omega_0\, e^{-at} \left[\cos \rho t - \frac{a}{\rho} \cdot \sin \rho t\right]$$

$$+\ \omega_2\, e^{-a(t-t_1)} \left[\cos \rho\,(t - t_1) - \frac{a}{\rho} \cdot \sin \rho\,(t - t_1)\right]. \tag{10}$$

Dorn points out that after the second impulse at $t = t_1$, the motion is the same as it would have been if there had been no such impulse, but if when $t = 0$, the values of θ and ω had been

$$-\frac{\omega_2}{\rho} \cdot e^{at_1} \cdot \sin \rho t_1, \tag{11}$$

and

$$\omega_0 + \omega_2 \cdot e^{at_1} \left[\cos \rho t_1 + \frac{a}{\rho} \cdot \sin \rho t_1\right], \tag{12}$$

and shows that the formulas can easily be generalized to fit the case in which there are a number of belated impulsive changes in the angular velocity, instead of one.

In the motion represented by (3) and (4), the angular velocity vanishes at the time t' defined by the equation

$$\tan \rho t' = \frac{\rho \omega'}{\alpha \omega' + \beta^2 \theta'}, \tag{13}$$

and if the first root be used, the amplitude at the first elongation is

$$e^{-\alpha t'} \left[\theta' \cdot \cos \rho t' + \frac{\omega' + \alpha \theta'}{\rho} \cdot \sin \rho t'\right]. \tag{14}$$

For the motion defined by (5), (6), (9), and (10), therefore, the first amplitude can be found by substituting for θ' and ω' in (13) and (14) the values given by (11) and (12). The computation is, however, not very simple, and we shall do well to treat the matter graphically, using equation (9) as the basis of our work.

If we define the function $F(t)$ by the equation

$$F(t) = e^{-\alpha t} \sin \rho t \tag{15}$$

and denote the constants $\frac{\omega_0}{\rho}$, $\frac{\omega_2}{\rho}$ by p and q, (9) may be written in the form

$$\theta = p \cdot F(t) + q \cdot F(t - t_1). \tag{16}$$

For any given galvanometer with a given resistance of the coil circuit α and ρ are definite, easily determined constants, and $F(t)$ is therefore determined. For the galvanometer represented by Figure 1, Plate 1, for instance, ρ is twice α for a coil circuit resistance of about 150 ohms. If we represent ρt by x, ρt_1 by x_1, and the ratio of α to ρ by μ, then

$$\theta = p \cdot e^{-\mu x} \sin x + q \cdot e^{-\mu(x - x_1)} \sin (x - x_1) = p \cdot f(x) + q \cdot f(x - x_1). \tag{17}$$

If then we draw the curves $y = p \cdot f(x)$, $y = q \cdot f(x)$, the ordinates of which are in the constant ratio p/q, and displace the second curve bodily to the right through the distance x_1, the sum of the ordinates of the first curve and the displaced curve will represent θ. For most purposes only the ratio (r) of q to p is important, and in plotting the curves we may make $p = 1$ and q, r.

To illustrate the process just described, let us suppose that when the galvanometer coil is at rest in its position of equilibrium, an impulsive current is sent through it, and after the coil, in response to this impulse, has had about half time enough to reach its elongation, a second impulse is given it half as strong as the first. The general form of the diagram will be much the same whether the damping be

very slight or so strong that the motion is just aperiodic, but in Figure D the lines are drawn to scale for the case $a/\rho = 1/2$.

OEUD is the curve $y = e^{-x/2} \cdot \sin x$, which reaches its maximum at M. OPFC is the curve $y = \frac{1}{2} \cdot e^{-x/2} \cdot \sin x$, and AFB is the last curve moved to the right through the distance $x = \rho t_1$. The angular deviation of the coil is given as a function of ρt by the broken curve OEGH, the ordinates of which are the sums of the corresponding ordinates of OEMD and AFB. The maximum of this curve belongs to a point

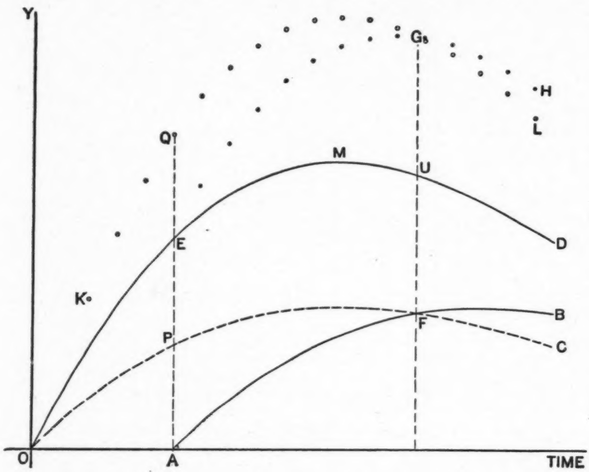

FIGURE D.

slightly to the left of G and measures the throw of the coil under the circumstances. If both impulses had been given to the coil when it was at rest, the deviation would have been given by the curve OKQGL. The actual throw is about 96 per cent of the throw which would be obtained if both impulses came together at the beginning. The actual values of a and ρ are not needed, and one does not need to know the period of the coil, the actual intensities of the impulses, or anything else, besides λ and r. In this case it is easy to find out by trial in two or three minutes how great the lag OA may be if the difference of the throws is not to be greater than one half per cent, for instance.

If the secondary of an induction coil which has no iron core be connected with the coil of the galvanometer represented by Figure 1, Plate 1, and if when the current I is running steadily through the primary

of the induction apparatus the primary circuit be first broken and then, after the coil has had just one quarter enough time to reach its elongation, closed in reverse direction, the angular deviation of the coil will be given as a function of ρt by the curve OBMVJC, Figure E. The ordinates of this curve are the sums of the corresponding ordinates of OBDL and ADK. If the current in the primary circuit of the induction apparatus were suddenly reversed while the galvanometer coil was at rest in its position of equilibrium, the deviation would be given by the

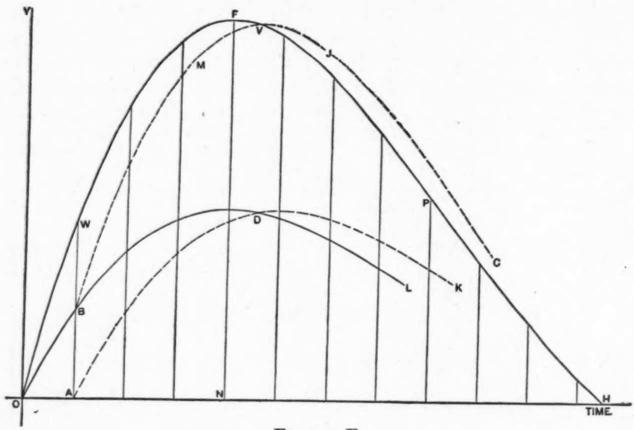

FIGURE E.

curve OWFPH, the ordinates of which are double those of the curve OBDL. The throw with the lag OA is nearly 99 per cent of that when the current is suddenly reversed.

This graphical process is especially convenient when the allowable decrease of throw is given and one wishes to find the maximum lag which will not make the throw difference too great. If the lag is given and the throw difference is wanted, this may be found by computation, though the graphical treatment has solid advantages. It is evident that the curve $y = e^{-\lambda x} \cdot \sin x$ serves for a given galvanometer with a given coil circuit for throws of all magnitudes.

It often happens that one has to work with a galvanometer the period of which is rather too short for the purpose in hand, but it is usually possible to determine, in the manner pointed out above, a correction factor to be applied to all throws, which will make the instrument trustworthy.

When a galvanometer is critically damped $\beta^2 = \alpha^2$, $\rho = 0$, and the equation of motion is

$$\frac{d^2\theta}{dt^2} + 2\,\alpha\,\frac{d\theta}{dt} + \alpha^2\theta = 0, \tag{18}$$

and the general solution of this is

$$\theta = (A + Bt)\,e^{-\alpha t}. \tag{19}$$

If when $t = 0$, $\theta = \theta'$, and $\omega = \omega'$;

$$\theta = [\theta' + (\omega' + \alpha\theta')\,t]\,e^{-\alpha t}. \tag{20}$$

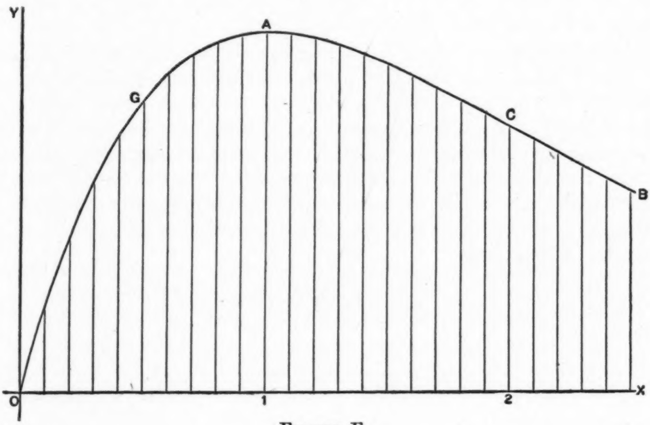

FIGURE F.

If when the coil is at rest in its position of equilibrium, an impulsive current sent through the instrument gives the coil an initial angular velocity ω_0,

$$\theta = \omega_0 \cdot t \cdot e^{-\alpha t}, \;\; \omega = \omega_0 \cdot e^{-\alpha t}\,(1 - \alpha t), \tag{21}$$

and if after this motion has gone on until the time t_1 a second impulse increases the angular velocity by the amount ω_2, then after the second impulse

$$\theta = \omega_0 t\, e^{-\alpha t} + \omega_2\,(t - t_1)\,e^{-\alpha\,(t - t_1)}. \tag{22}$$

It is possible to give to this equation also a graphical treatment

similar to that which we have discussed above for the case where α is less than β. If $\phi(t)$ is defined by the equation

$$\phi(t) = \alpha t\, e^{-\alpha t},$$

$$\theta = \frac{\omega_0}{\alpha} \cdot \phi(t) + \frac{\omega_2}{\alpha} \cdot \phi\,(t - t_1). \tag{23}$$

Figure F shows the form of the curve $y = xe^{-x}$.

In considering the magnitude of the throw of a damped ballistic galvanometer due to a given continuously varying current which flows through the coil for a finite time interval, we shall do well to use Dorn's results in nearly the forms into which they have been put by Diesselhorst in his important paper on the subject.

When the suspended system is at rest in its position of equilibrium, a short-lived current shall flow through the coil and shall have the intensity, I, which is a given function of the time. From the epoch $t = \tau$, I shall have the value zero. The product of the strength of the magnetic field between the poles of the permanent magnet, at the place where the coil is, and the effective area of the turns of the coil shall be denoted by q, so that while the current is flowing, the equation of motion of the coil, for such small angles as are used in mirror instruments, has the form

$$K \cdot \frac{d^2\theta}{dt^2} + 2\,a \cdot \frac{d\theta}{dt} + b^2 \cdot \theta = qI \tag{24}$$

or

$$\frac{d^2\theta}{dt^2} + 2\,\alpha \cdot \frac{d\theta}{dt} + \beta^2 \cdot \theta = \frac{q}{K} \cdot I = \mu \cdot I. \tag{25}$$

If, as before, m and n are the roots of the equation $x^2 + 2\,\alpha x + \beta^2 = 0$, if Q_t represents the whole flux of electricity through the coil from $t = 0$ to $t = t$, and if M_t, N_t represent the ratios to Q_t of the integrals

$$\int_0^t I \cdot e^{-mt} \cdot dt, \qquad \int_0^t I \cdot e^{-nt} \cdot dt, \tag{26}$$

respectively, then the solution of (25) is

$$\theta = \frac{\mu}{m - n}\,[e^{mt} \cdot \int_0^t I \cdot e^{-mt} \cdot dt - e^{nt} \int_0^t I \cdot e^{-nt} \cdot dt] \tag{27}$$

$$= \frac{\mu Q_t}{m - n}\,[M_t \cdot e^{mt} - N_t \cdot e^{nt}]. \tag{28}$$

After the time $t = \tau$, M_t, N_t have the constant values M, N, and Q_t becomes Q, the total amount of electricity carried by the current from $t = 0$ until it ceases to flow at $t = \tau$, so that

$$\theta = \frac{\mu Q}{m - n}\,[M \cdot e^{mt} - N \cdot e^{nt}]. \tag{29}$$

If, as is usually true in practice, β is greater than a, ρ is positive, $m = -a + \rho i$, $n = -a - \rho i$, $m/(m - n) = \frac{1}{2} + ai/2\rho$, $n/(n - m) = \frac{1}{2} - ai/2\rho$, but the results are, of course, real.
If we determine $d\theta/dt$ from (29) and equate it to zero, we learn that at a time of elongation

$$t = \frac{1}{m - n} \cdot \log\left(\frac{nN}{mM}\right), \tag{30}$$

and this value of t substituted in (29) gives the amplitude at elongation in the form

$$A = \frac{\mu Q}{m - n}\left[\left(\frac{n}{m}\right)^{\frac{m}{m-n}} - \left(\frac{n}{m}\right)^{\frac{n}{m-n}}\right] M^{\frac{n}{n-m}} \cdot N^{\frac{m}{m-n}} \tag{31}$$

$$= C \cdot M^{\frac{n}{n-m}} \cdot N^{\frac{m}{m-n}} \tag{32}$$

where C is a function of the constants of the galvanometer and is independent of the manner in which the whole flux Q of electricity is sent through the circuit. If A_0 denote the amplitude at the first elongation when Q is sent impulsively through the coil at the origin of time,

$$\frac{A}{A_0} = M^{\frac{n}{n-m}} \cdot N^{\frac{m}{m-n}}. \tag{33}$$

If I happens to be given in analytic form as a function of t, it is possible, as Diesselhorst shows in a general case, to obtain a convergent series for A/A_0. For the purposes of this paper, however, where the form of I is shown merely by an oscillograph record, we shall find it desirable if m and n are real, to plot the curves $y = Ie^{-mt}$, $y = Ie^{-nt}$ directly from this record and then to find the values of M and N by mechanical integration.

If β is greater than a, (27) may be written

$$\theta = \frac{\mu}{\rho}\,e^{-at}\left[\sin \rho t \cdot \int_0^t I \cdot e^{at} \cdot \cos \rho t \cdot dt - \cos \rho t \cdot \int_0^t I \cdot e^{at} \cdot \sin \rho t \cdot dt\right]. \tag{34}$$

$$\text{If } R \cdot Q = \int_0^\tau I \cdot e^{at} \cdot \cos \rho t \cdot dt \text{ and } S \cdot Q = \int_0^\tau I \cdot e^{at} \cdot \sin \rho t \cdot dt, \tag{35}$$

the value of θ after the current has ceased is

$$\theta = \frac{\mu Q e^{-at}}{\rho} [R \cdot \sin \rho t - S \cos \rho t] \tag{36}$$

where Q, R, and S are constants.

At the first elongation,

$$\tan \rho t = \frac{\rho R + a S}{a R - \rho S} \tag{37}$$

or

$$\cos \rho t = \frac{a R - \rho S}{\beta \sqrt{R^2 + S^2}} \qquad \sin \rho t = \frac{\rho R + a S}{\beta \sqrt{R^2 + S^2}}, \tag{38}$$

and if the first root of these equations be substituted for t in (36), it appears that the first elongation is given by the expression

$$A = \frac{\mu Q \sqrt{R^2 + S^2}}{\beta} \cdot e^{-u} \tag{39}$$

where

$$u = \frac{a}{\rho} \cdot \tan^{-1} \frac{\rho R + a S}{a R - \rho S}. \tag{40}$$

If the quantity Q of electricity had been sent impulsively through the galvanometer when the coil was at rest in the position of equilibrium, the throw would have been as (5) shows

$$A_0 = \frac{\mu Q}{\beta} \cdot e^{-v} \tag{41}$$

where $v = \frac{a}{\rho} \cdot \tan^{-1} \frac{\rho}{a}$.

Hence

$$\frac{A}{A_0} = \sqrt{R^2 + S^2} \cdot e^{-(u-v)} = \sqrt{R^2 + S^2} \cdot e^{-w}, \tag{42}$$

where $w = \frac{a}{\rho} \cdot \tan^{-1} \frac{S}{R}$.

If $\frac{1}{2}\, Q$ were sent impulsively through the circuit at $t = 0$, and $\frac{1}{2}\, Q$ at $t = \tau$, the values of R and S to be used in (42) would be

$$R = \tfrac{1}{2} (1 + e^{a\tau} \cdot \cos \rho\tau), \quad S = \tfrac{1}{2}\, e^{a\tau} \cdot \sin \rho\tau. \tag{43}$$

With some of the forms of short period, critically damped d'Arsonval galvanometers commonly used in American laboratories, it is difficult to reverse the current in the primary of an induction apparatus with air core by a large double throw switch so quickly as to avoid a decrease in the throw of the galvanometer coil owing to the lag in the second impulse.

If a current of constant intensity (Q/τ) flowing for the time interval τ conveys a quantity, Q, of electricity through the circuit, the values of R and S are

$$R = \frac{1}{\beta^2 \tau} \left[e^{a\tau} (\rho \cdot \sin \rho\tau + a \cdot \cos \rho\tau) - a \right] \tag{44}$$

$$S = \frac{1}{\beta^2 \tau} \left[e^{a\tau} (a \sin \rho\tau - \rho \cdot \cos \rho\tau) + \rho \right] \tag{45}$$

$$\sqrt{R^2 + S^2} = \frac{1}{\beta\tau} \sqrt{e^{2a\tau} - 2\, e^{a\tau} \cos \rho\tau + 1}. \tag{46}$$

In the case of a critically damped instrument

$$\theta = \mu e^{-} \ \left[t \int_0^t I \cdot e^{at} \cdot dt - \int_0^t I \cdot t \cdot e^{at} \cdot dt \right].$$

If there were no damping, a would be zero, e^{-u} would be equal to unity, and R and S would satisfy the equations

$$RQ = \int_0^\tau I \cdot \cos \rho t \cdot dt, \qquad SQ = \int_0^\tau I \cdot \sin \rho t \cdot dt.$$

The foregoing theory rests, of course, upon the assumption that the swinging system of a galvanometer meets with a resistance to its motion which may be attributed to a force couple of moment equal at any instant to the product of a fixed constant and the angular velocity which the system then has. It is evident, however, that this condition cannot be exactly fulfilled during the whole motion of the needle or coil of any instrument in which the damping soon brings the swinging system absolutely to rest. In the case of a horizontal bar magnet swinging without sensible friction about a vertical axis through its centre, the ratio of successive half amplitudes usually remains nearly constant for a large portion of the motion, though the actual value of the ratio often depends upon the atmospheric conditions, as Gauss showed. The logarithmic decrement of the oscillations of a magnetic

needle swinging in a strong field under the damping action of a mica vane of the usual kind usually diminishes as the amplitudes grow smaller. The same tendency often shows itself in the case of a d'Arsonval galvanometer when the damping, either electromagnetic or atmospheric, is fairly large.

In a galvanometer of any of the common forms in which the restoring moment is due, not to the mutual action of a magnet and the external field, but to torsional forces in a spring or suspending fibre, even though the system comes to rest sensibly at its old position of equilibrium, the swings are often one-sided in a fashion best described, perhaps, with the help of an example or two.

A certain d'Arsonval galvanometer (Y) of the Ayrton and Mather type was connected in series with a rheostat of resistance R and the coil of a small magneto-inductor. The period of the galvanometer coil was dependent of course upon the value of R: when the circuit was broken, its value was about 16.5 seconds. The same flux change in the coil of the inductor might be made over and over again at pleasure by slipping the coil in one direction or the other between two fixed stops. The resistance of the galvanometer and the inductor coil together was about 96.6 ohms. When the galvanometer coil was at rest in its position of equilibrium (scale reading 711), and the value of R was 600 ohms, the inductor coil was moved quickly from one stop to the other and a short series of turning points, 329, 886, 623, 750, 689, were observed. When the inductor coil was slipped back to its original place, the readings were 1095, 534, 799, 672, 733. Using the first set of turning points and the zero 711, the successive half amplitudes were 382, 175, 88, 39, 22, and the ratios of the successive pairs were 2.18, 1.99, 2.26, 1.77. The other set of turning points give the half amplitudes 384, 177, 88, 39, 22, and the ratios, 217, 2.01, 2.26, 177. The half sums of corresponding numbers in the two observed sets are 712, 710, 711, 711, 711, and there is no obvious bias in favor of deflections on one side of the zero point. There was no sensible "set" when the system came to rest, but during the swings there seemed to be a very slight movement of the zero point towards the side of the first excursion, at the end of which the whole angle of twist in the long gimp was only about 1°. When R was made 400 ohms, the time of swing fell from 8.6 seconds to 8.2 seconds, the throw due to the same movement of the inductor coil rose to 483, and the ratios of successive pairs of half amplitudes became 3.16, 2.68, 3.17. When the twist in the gimp per centimeter of its length is made as large as in many of the instruments in common use, the tendency here noted becomes very troublesome, and it is difficult to determine from a short set of throws corresponding to

a fairly strong damping what the value of the logarithmic decrement should be.

A certain d'Arsonval galvanometer (X), of the type represented in Figure C, which was formerly in use in the Jefferson Laboratory, had a period of 149 seconds. When the coil was given a deflection corresponding to a scale reading of 14.15 cms., and was then allowed to swing, the ratios of the successive half amplitudes were 1.066, 1.061, 1.067, 1.061, 1.066, 1.060, etc.

TABLE I.

R.	T.	ρ.	λ.	α.	β.
3000	7.00	1.030	1.207	0.396	1.104
4000	5.95	1.056	0.699	0.234	1.082
10000	5.78	1.086	0.398	0.137	1.096
20000	5.74	1.094	0.224	0.128	1.097
Infinity	5.73	1.097	0.032	0.011	1.097

The galvanometers (X, Y) just mentioned, unlike most of those which are usually available in a laboratory, were almost exactly symmetrical in their throws on opposite sides of the zero. In most large instruments in which the coils are wound on open metal frames, there is a slight bias, so that a given flow of electricity sent impulsively through the circuit causes a little larger throw on one side than on the other. Sometimes the bias, when the always small throw is increased by increasing the discharge, changes sign; sometimes levelling the instrument will help a trifle, but usually the lack of symmetry seems to be connected with the magnetism induced in the frame or the coil by the field of the magnet.

Mr. John Coulson, who has studied in the Jefferson Laboratory the characteristics of an excellent short period d'Arsonval galvanometer of the very best make, has found a bias of about 2 per cent in favor of the throws on one side of the zero point. In this instrument there is also the same irregularity in the ratios of successive amplitudes which has been already noticed. For a given impulse, which caused a throw on one side, after which the coil oscillated with decreasing amplitude, the ratios were 2.16, 2.03, 2.15, 2.08, while the same impulse reversed in direction gave the ratios 2.09, 2.12, 2.09, 1.97. These values were persistent and could be obtained over and over, and their differences were quite large

enough to disturb a person who is attempting to get an accurate value of the so-called damping coefficient for use in the differential equation.

Some of the constants of this galvanometer as determined by Mr. Coulson are given in Table I.

Such slight departures from symmetry as these seem, however, not to affect in the least the usefulness of a good d'Arsonval galvanometer in measuring quantities of electricity sent through its coil; the mean of throws on opposite sides of the zero point due to a given impulsive discharge remains practically constant, and a good calibration might often be made to serve for a long time, though the instrument should be tested, of course, every time it is used.

In view of the fact that the motion of the coil of a d'Arsonval galvanometer usually deviates somewhat, as we have seen, from the course laid down by the Gaussian theory, we may inquire whether such equations as (14), (33), (42), based on that theory, agree with the results of observations on ordinary instruments. It may be well to say at the outset that, according to my experience, the agreement is wonderfully close.

To support this assertion I may adduce first a simple test made a long time ago upon the galvanometer X mentioned above. If we assume for λ the value 0.0611, the natural logarithm of 1.063, and for T the value 149, it appears that $a = 0.00082$ and $\rho = 0.0422$. The time required for the swing out from the zero to the turning point is then $\frac{1}{\rho}\tan^{-1}\left(\frac{\rho}{a}\right)$ or 36.4 seconds: the return to the zero requires 38.1 seconds. If under these circumstances a given impulse be sent through the coil, and after an interval $\tau = 10$ seconds, another equal impulse, the resulting throw should bear to that which would be caused if both impulses came together at the beginning, the ratio given by (42) when $a\tau = 0.082$, and $\rho\tau = 0.422$, which corresponds to 24.18°. In this case $R = 0.9597$, $S = 0.2064$, $\sqrt{R^2 + S^2} = 0.982$, $\log e^{-w} = 9.9980$, and A/A_0 is about $0.977 +$. Now when a single impulse from an induction apparatus without iron was sent through the coil, and after a delay of ten seconds another equal to the first, the throw as given by a number of readings was 1144, but the reading when both came together was 1170. The ratio of these numbers is 0.978. It is easy to show by a little computation that if the delay were 5 seconds, the ratio of A to A_0 would be 0.994; but if it were 30 seconds, the ratio would be about 0.806.

TABLE II.

$R.$	∞	500.	300.	200.	100.	50.	30.	0.
T	132.0	132.1	132.4	132.8	133.7	136.0	137.8	143.0
r	1.23	1.34	1.40	1.46	1.68	1.94	2.19	3.12
λ	0.207	0.293	0.336	0.378	0.519	0.663	0.784	1.138
$\alpha = 2\lambda/T$	0.00314	0.00443	0.00508	0.00570	0.00776	0.00975	0.01138	0.01592
$\rho = 2\pi/T$	0.0476	0.0476	0.0474	0.0473	0.0470	0.0462	0.0456	0.0439
α/ρ	0.066	0.093	0.107	0.120	0.165	0.211	0.250	0.362
β	0.0477	0.0477	0.0477	0.0476	0.0476	0.0472	0.0470	0.0468

Table II gives some of the results of several days' study of the characteristics of the galvanometer V. The periodic time, which was determined with the help of a chronograph, is given in round numbers, because slight differences of dampness in the air or of barometric pressure seemed to affect the period somewhat. With small values of R, the ratio (r) of successive half amplitudes was usually somewhat variable in the manner described above, though the values were persistent. Under these circumstances the average value is given. If the instrument followed the Gaussian law exactly, the value of β should be the same throughout.

As this galvanometer was to be used in an important series of magnetic measurements during which it was necessary to determine with accuracy the change of flux in the solid core of a fairly large electromagnet when the exciting current should be reversed in direction, it was desirable to study with some care the effect upon the throw due to the duration of the induced currents. If under all ordinary cases the area beneath the curve in the record of an oscillograph in series with the galvanometer is proportional to the corresponding throw of the galvanometer, one may assume that the performance of the galvanometer will continue to be satisfactory; but this test is not easy to make. It is

comparatively easy, however, to give to the galvanometer coil, by aid of a large induction apparatus with air core, such a series of given impulses at given time intervals as shall give all necessary information. In fact the simple device of determining the throw due to two equal impulses separated by the interval τ for a number of different values of τ will

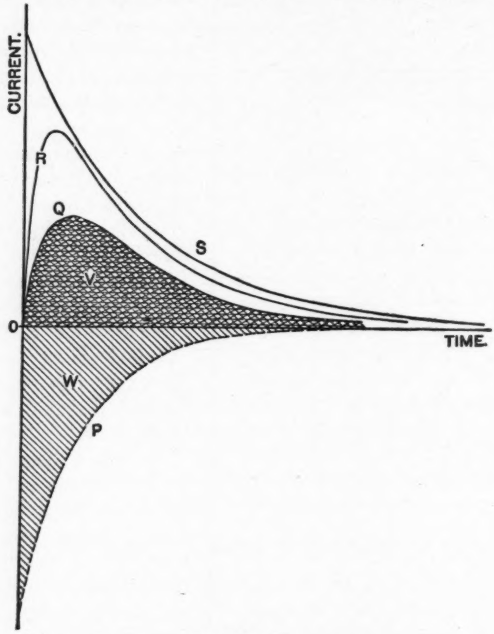

FIGURE G.

The curves Q, R, S represent for different relative values of the mutual inductance the current induced in the secondary circuit of a certain induction coil without iron, when the primary circuit is suddenly closed.

usually serve to decide sharply whether or not the galvanometer coil follows the Gaussian law closely enough to make it possible to predict its behavior under ordinary circumstances from the equations proved above. This kind of experiment was made with Galvanometer V: an adjustable commutator, driven through a train of wheels by a motor running very steadily at just under 30 revolutions per second, served to give the impulses at the right time interval apart. A series of

careful observations showed that the throw was 1471, 1470, 1468, 1464, 1458, 1452, 1444, according as the interval between the impulses was 0, 1, 2, 4, 6, 7, or 8 seconds. At this circuit resistance, $T = 139$, $\rho = 0.0450$, $a = 0.0125$, and if we assume the interval to be 8 seconds, $a\tau = 0.1$, and $\rho\tau = 0.360$, which corresponds to 20.63°. According to (43) under these conditions, $R = 1.017$, $S = 0.195$, $\sqrt{R^2 + S^2} = 1.035$, $\tan^{-1}(S/R) = 0.1891$, and $A/A_0 = 0.982$. That is, the throw when the second impulse follows the first at the interval of eight seconds should theoretically be only 982 thousandths of the throw due to the

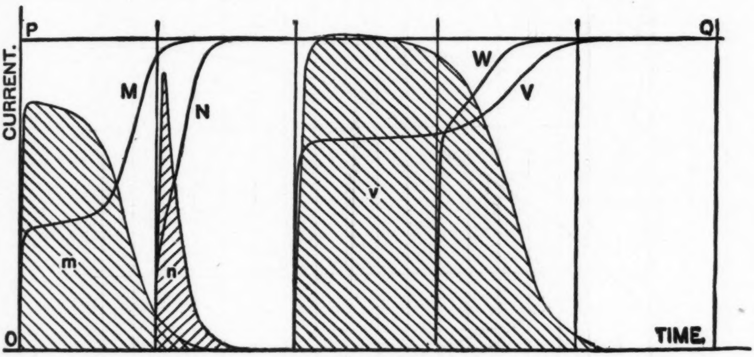

FIGURE H.

two impulses coming together. The results of experiment give 1444/1417 or 0.982. This exact coincidence is, of course, a matter of chance.

When the interval is 4 seconds, $a\tau = 0.05$, $\rho\tau = 0.180$, and $A/A = 0.995$; here again the agreement with observation is exact for $1464/1471 = 0.995$. For an interval of 6 seconds, theory gives for A/A_0 the value 0.992+ and experiment, 0.992−, so that the experimental results, which were obtained long before any computations were made, point to a complete agreement, within the limits of observation, with theory.

With this damping, corresponding to a value for R of about 25 ohms, the time required for the coil to reach its elongation from the zero point is about 28.9 seconds; the return takes 40.6 seconds. When R is 500, the time from the zero is 32.9 seconds, and the time back is 33.1 seconds.

When the circuit of the exciting coil of a large electromagnet is suddenly broken, the induced current in a test coil wound around the core rises very quickly to a maximum value and then falls away gradually: indeed the form of the current is usually much like that in the secondary circuit of an induction coil with air core when the primary current is suddenly interrupted. Such a current is shown by curve P of Figure

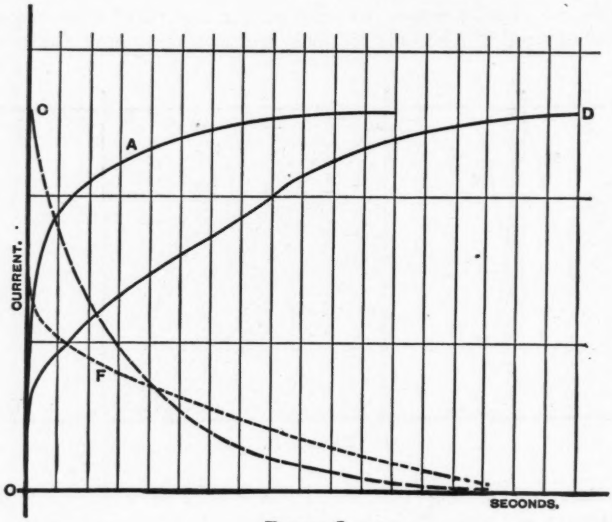

FIGURE I.

G, which is drawn for the case $M = L/2$ when the self-inductances of the two circuits are equal. If, after the current in the exciting coil of an electromagnet has been running steadily, its circuit be broken and after a short interval closed again, the induced current in the test coil will be very different according to the direction of the current in the main circuit. If the new direction is the same as that of the current before the break, the new current is called "direct," but if the new direction is opposed to the old, the new current is said to be "reversed." The curves M, N in Figure H, which are reproduced to scale from the records of an oscillograph, show the manners of growth of reversed and direct currents, respectively, in the exciting circuit of a certain electromagnet; and the boundaries of the shaded portions of the diagram show the forms of the induced currents. The shaded areas give the

whole transfer of electricity in the induced currents in the two cases. Besides the exciting coil, this magnet had another similar coil wound

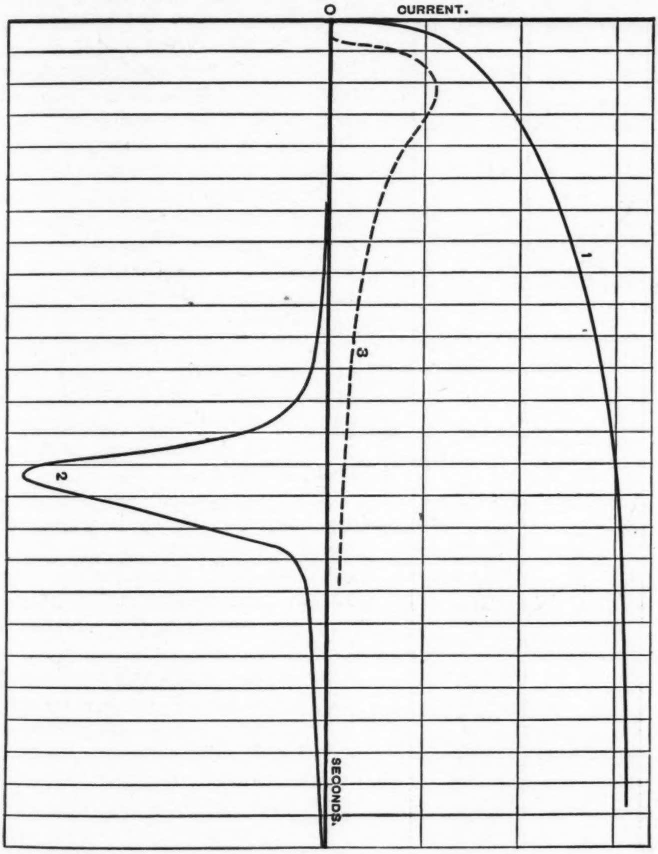

FIGURE J.

about the core. Curves V and W show the growth of reversed and direct currents in the exciting circuit when the last named coil was closed on itself, and the currents induced in it hindered the establishment of the

main current. The scale of the oscillograph in the secondary circuit was different from that used before, but the general shape of the induced current is shown by the boundary of the shaded area v. Curves C and F of Figure I show the forms of induced currents in the testing coil in the case of a very large magnet the cross section of the solid core of which had an area of about 500 square centimeters. A and D show the corresponding currents in the main circuit: in the first case the generator was a battery of 40 storage-cells, and a considerable amount of extra resistance was used in the circuit; in the second case the same final current was caused by a battery of 10 cells, and very little extra resistance was needed. This particular engraving, which was made by the "Wax Process," does not reproduce the original exactly, for the upper portions of A and D are here too nearly horizontal.

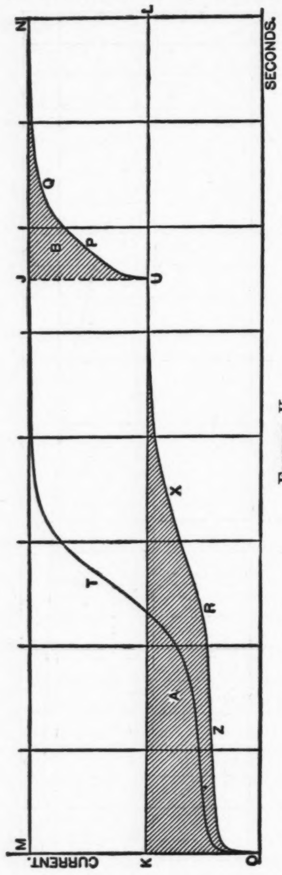

Figure K.

A very uncommon form of secondary current is shown in Figure J. Curve 1 represents the form of the main current of a very large electromagnet with massive core. At the axis of a portion of the core was a longitudinal hole about an inch in diameter, and in this hole was inserted an iron rod around which a layer of insulated wire was wound to serve as a test coil. Curve 2 shows the form of the induced current in this coil when the main circuit was closed; the dotted curve gives the form of the induced current when the main circuit was suddenly broken. The crest of the curve 2 does not come until fourteen seconds after the main current starts.

Figure K shows the manner of growth of a current of final intensity 2.3 amperes, under a voltage of perhaps 60, in a coil of 1388 turns

about the core of the magnet depicted in Figure A. The curve OTJN is a copy of the record of an oscillograph in the circuit when the electromotive force was suddenly applied at $t=0$. The area between this curve and its asymptote up to any value of the time represents the whole change of the flux of magnetic induction through the coil, and the difference between the ordinate of the asymptote and that of the curve is proportional to the instantaneous rate of change of this flux, and, therefore, to the induced electromotive force in a test loop

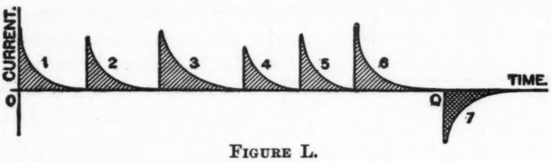

FIGURE L.

A portion of the record of an oscillograph in the circuit of a secondary coil wound on the core of an electromagnet when the current in the exciting coil is made to change by sudden steps in the determination of a hysteresis cycle.

passed around the core. The general form of the induced current in such a secondary circuit might be seen by looking at the curve just mentioned upside down and through the paper. In this case the induced current would practically come to an end in about five and one half seconds. The line OZRXUPQN shows the growth of the main current when there was an extra non-inductively wound resistance in the circuit which was suddenly shunted out after about five and one half seconds. Here, again, the general shape of the induced current in the secondary circuit might be seen by looking at this line upside down, from behind. The intensity of the induced current was inappreciable after about eight seconds.

Figure L shows the general shape of the induced currents in the circuit of a test coil of a few turns wound on the core of an electromagnet when the current in the exciting circuit is made to grow by shunting out a part of the resistance of this circuit by steps. If the currents, up to the time OQ were sent through the coil of a long period ballistic galvanometer, the resulting throw would not fall so much below the throw due to the whole quantity of electricity carried by the currents, sent instantaneously through the galvanometer at the origin of time, as would the throw due to a steady current lasting for the time OQ and carrying the same total amount.

The examples already given will serve well enough to show what is required of a galvanometer which shall measure accurately the whole

quantity of electricity which flows in the test coil. Of course, the induced current may last with an extremely feeble intensity for a long time, but in any practical case it is easy to set a limit of time after which no sensible flow will occur.

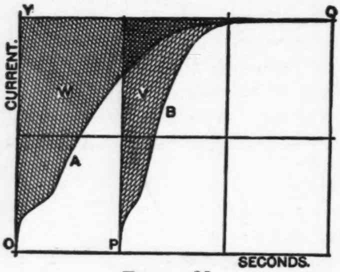

FIGURE M.

Figure M shows two reverse current curves for a toroidal magnet. The final strength of the current was the same in both cases, but the applied electromotive force was twice as great in the case of the curve B as in the case of the curve A.

If A_0 is the throw which would be caused by an instantaneous discharge of Q units of electricity through a galvanometer at the beginning of motion, A' the throw caused by an instantaneous discharge of $\frac{1}{2}$ Q units at the beginning and another discharge of $\frac{1}{2}$ Q units seconds later, and A'' the throw due to a steady current of Q/τ units intensity lasting from $t = 0$ to $t = \tau$, then A' is less than A'', and this in turn is less than A. Occasionally one encounters an induction current which has a form much like that indicated in Figure N by the curve KLG, and we shall find it interesting to determine the ratio A''/A for one or two practical cases. It is well to notice that the second member of (42) depends only upon the ratios $\lambda = a/\rho$ and

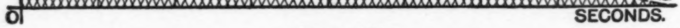

FIGURE N.

$\delta = \tau/T$, and not at all upon the other constants of the instrument; for if we write $z = \rho t$ and $I = f(t) = \phi(z)$, we shall find that

$$R = \frac{\int_0^{2\pi\delta} \phi(z) \cdot e^{\lambda z} \cdot \cos z \cdot dz}{\int_0^{2\pi\delta} \phi(z) \cdot dz}, \quad S = \frac{\int_0^{2\pi\delta} \phi(z) \cdot e_{\lambda z} \cdot \sin z \cdot dz}{\int_0^{2\pi\delta} \phi(z) \cdot dz}, \quad (47)$$

and these expressions involve λ and δ but are independent of the sensitiveness of the galvanometer and of its time of swing.

It is possible to show from equations (44) and (45), after some computation, that for the case of the galvanometer V, for which we may take $\alpha = 0.0125$, $\rho = 0.0450$; $A''/A_0 = 0.994$, or 0.998, according as τ is 8 seconds or 4 seconds. It is well to recall the fact mentioned above, that $A'/A = 0.982$ or 0.995, according as $\tau = 8$ seconds or 4 seconds.

Perhaps most of the induction currents which one meets in making magnetic measurements have forms similar to those of the curves S or P in Figure G, and it is worth while to compute the value of the ratio A/A_0 on the supposition that the current flows from $t = 0$ to $t = \tau$ with the intensity $I = k\ (\tau - t)$ where it is clear that $k = 2Q/\tau^2$.

Since

$$\int x \cdot e^{\lambda x} \cdot \sin x \cdot dx = \frac{e^{\lambda x}}{(1+\lambda^2)^2}\left[(\lambda \cdot \sin x - \cos x)(\lambda^2 x + x - \lambda)\right.$$
$$\left. + (\sin x + \lambda \cdot \cos x)\right], \quad (48)$$

and

$$\int x \cdot e^{\lambda x} \cdot \cos x \cdot dx = \frac{e^{\lambda x}}{(1+\lambda^2)^2}\left[(\sin x + \lambda \cdot \cos x)(\lambda^2 x + x - \lambda)\right.$$
$$\left. - (\lambda \cdot \sin x - \cos x)\right], \quad (49)$$

it is not difficult to prove that when $I = k\,(\tau - t)$,

$$R = \frac{2}{\beta^4 \cdot \tau^2}\left[\alpha \cdot e^{\alpha\tau}(\rho \cdot \sin \rho\tau + \alpha \cdot \cos \rho\tau)\right.$$
$$\left. + \rho \cdot e^{\alpha\tau}(\alpha \cdot \sin \rho\tau - \rho \cdot \cos \rho\tau) + \rho^2 - \alpha^2 - \alpha\beta^2\tau\right], \quad (50)$$

$$S = \frac{2}{\beta^4 \cdot \tau^2}\left[\alpha \cdot e^{\alpha\tau}(\alpha \cdot \sin \rho\tau - \rho \cdot \cos \rho\tau)\right.$$
$$\left. - \rho \cdot e^{\alpha\tau}(\rho \cdot \sin \rho\tau + \alpha \cdot \cos \rho\tau) + \beta^2\rho\tau + 2\,\alpha\rho\right]. \quad (51)$$

These formulas are not very well adapted for easy computation, and in many practical cases in which the quantities in the brackets are very small and the coefficient $2/\beta^4\tau^2$ very large it is desirable to use five or six place logarithms in the work. As an illustration of the use of these equations we may consider the instance of the galvanometer V through which a current of the form $I = k\ (\tau - t)$ shall flow for 8 seconds. Here $\alpha = 0.0125$, $\rho = 0.0450$, $\beta^2 = 0.0021812$, $2/\beta^4\tau^2 = 6568.39$, $R = 1.04723$, $S = 0.12545$, and $A/A_0 = 0.9974$. The throw due to this current is the same within about one quarter of one per cent as if the whole amount of electricity conveyed by the cur-

rent had been sent instantaneously through the coil at the time $t = 0$. For a galvanometer of the same period with practically no damping the value of A/A_0 under the circumstance just mentioned would be about 0.9964. A current of the form $I = k(\tau - t)$ and lasting for 34 seconds would, in the case of the galvanometer W, give a throw within about one third of one per cent the same as an impulsive discharge of the same total amount would cause if sent through the coil at the origin of the motion.

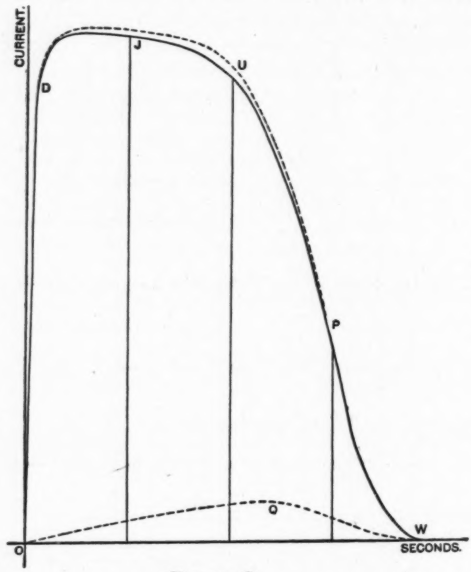

FIGURE O.

For a current of the general shape of S (Figure G) regarded as stopping at the time $t = \tau$, the ratio of A/A_0 would be much more nearly unity than for a current of the form $I = k(\tau - t)$.

If as in the case of an induction coil without iron, when the primary circuit is suddenly broken, I is of the form $I_0 \cdot e^{-kt}$, and if we write $g = a - k$,

$$RQ = \frac{I_0}{g^2 + \rho^2}[e^{g\tau}(\rho \cdot \sin \rho\tau + g \cdot \cos \rho\tau) - g], \tag{52}$$

$$SQ = \frac{I_0}{g^2 + \rho^2}[e^{g\tau}(g \cdot \sin \rho\tau - \rho \cdot \cos \rho\tau) - \rho], \tag{53}$$

$$Q = \frac{I_0}{k}(1 - e^{-k\tau}). \tag{54}$$

If $g = -\frac{1}{2}$, $\alpha = 0.0125$, and $\rho = 0.0450$; the value of A/A_0 will be 0.989, if the current flows until the needle reaches its elongation, say for 29 seconds.

When the shape of an induced current which is to pass through a ballistic galvanometer of long period is not analytically simple, it is always possible to determine by mechanical integration, with sufficient accuracy, the ratio of the throw caused by the current to the throw which the same total quantity of electricity sent instantaneously through the instrument would give. As an example, we may consider the form of current represented by the curve ODJPW of Figure O, which is a fairly close copy of an oscillogram. If we assume that the duration of the current is to be 4 seconds and that galvanometer V is to be used, so damped that

$$\alpha = 0.0125, \qquad \rho = 0.0450,$$

it is easy to measure a number of ordinates of the current curve, multiply each by the corresponding values of $e^{\alpha t} \cdot \cos \rho t$, $e^{\alpha t} \cdot \sin \rho t$, and thus compute the ordinates of the curves OUPW and OQW. The areas under these curves obtained by a good planimeter represent RQ and SQ of (35) and (42), and the area under the current curve gives Q on the same scale. An actual trial would show that A falls below A_0 by about one seventh of one per cent. If the galvanometer W were used, it would be quite impossible to detect the difference between A_0 and A, even if the duration of the current, of the form shown, were as much as 16 seconds.

The galvanometers V and W are to be used in making determinations by the "Isthmus Method" of the ultimate values of the intensity of magnetization in a large number of specimens of magnetic metals, in cases where it is necessary to reverse the direction of the exciting currents. When a rather small yoke which weighs about 300 kilograms is used under a fairly high voltage, V works very well: the whole duration of the induced current is practically less than 5 seconds, and the intensity falls off rapidly after the first, so that the difference between A and A_0 is wholly inappreciable. For very high values of the induction a solid yoke of the form shown in Figure B is to be employed. In this case the smallest cross section of the core has an area of 450 square centimeters, and it is not possible sensibly to reverse an excitation of

say one hundred and fifty thousand ampere turns about this core in less than about 30 seconds under any practicable voltage. Of course the process is not completed even in this time, but the amount of electricity carried by the induced current after 30 seconds can be made relatively very small. Indeed for the shape of current practically encountered with this apparatus, the duration of the flow might be 60 seconds without causing a decrease of more than a fraction of one per cent in the throw of the galvanometer W.

I wish to express my obligation to the Trustees of the Bache Fund of the National Academy of Sciences for the loan of apparatus used in studying for this paper some of the induction current diagrams.

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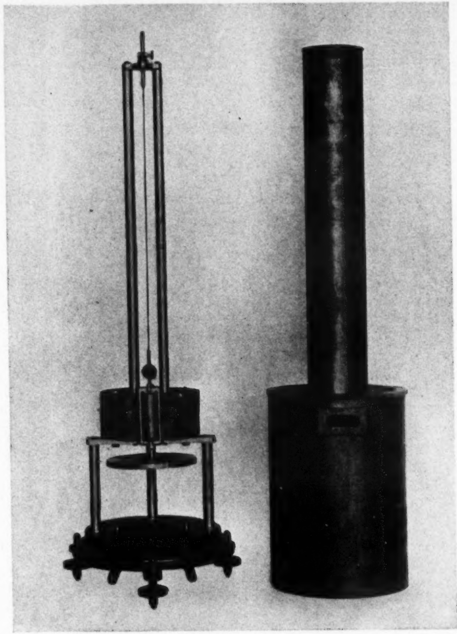

FIGURE 1.

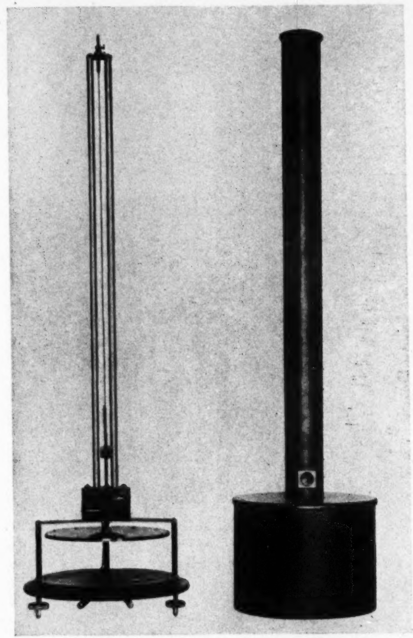

FIGURE 2.